



1.6.8.9 The square root of complex numbers

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Introduction

- In this topic, we will
 - Review the square root of real numbers
 - Find the square roots of a complex number
 - Define one to be the *principal* square root
 - Look at a geometric interpretation
 - Observe a property





Review

- Note that if $x > 0$, then

$$\left(\sqrt{x}\right)^2 = \left(-\sqrt{x}\right)^2 = x$$

- Similarly,

$$\left(\sqrt{x}j\right)^2 = \left(-\sqrt{x}j\right)^2 = -x$$

- Question: What is $\sqrt{3+j}$?





Square root of a complex number

- Solution:

- If $z = |z| \angle \theta$, then $w = \sqrt{z}$ if $w^2 = z$

- Suppose $w = |w| \angle \phi$:

- If $w^2 = (|w| \angle \phi)^2 = |w|^2 \angle (2\phi) = |z| \angle \theta = z$,

then $|w|^2 = |z|$ and $2\phi = \theta$

- Therefore, $w = \pm \sqrt{|z|} \angle \frac{\theta}{2}$





Square root of a complex number

- We will define $\sqrt{z}$ to be
 - That root with the greatest real component, and
 - If real components are equal,
the root with the greatest imaginary component
- The easiest is in the polar representation:
 - Require that $-\pi < \theta \leq \pi$, in which case

$$\sqrt{z} \stackrel{\text{def}}{=} \sqrt{|z|} e^{i\frac{\theta}{2}}$$





Square root of a complex number

- Note that our definition still works for real numbers:
 - If z is a positive real, then $z = \alpha + 0j = \alpha \angle 0$,
thus

$$\sqrt{z} = \sqrt{\alpha} \angle \frac{0}{2} = \sqrt{\alpha} \angle 0 = \sqrt{\alpha}$$

The other root is

$$-(\sqrt{\alpha} + 0j) = -\sqrt{\alpha}$$





Square root of a complex number

- Also, it works for negative numbers
 - If z is a negative real, then $z = -\alpha + 0j = \alpha \angle \pi$,
thus

$$\sqrt{z} = \sqrt{\alpha} \angle \frac{\pi}{2} = \sqrt{\alpha} j$$

The other root is

$$-\left(0 + \sqrt{\alpha} j\right) = -\sqrt{\alpha} j$$





Square root of a complex number

- Examples:

- If $z = 1 + j = \sqrt{2} \angle \frac{\pi}{4}$,

$$\sqrt{z} = \sqrt{1 + j} = \sqrt{\sqrt{2}} \angle \left(\frac{\pi}{4} / 2 \right)$$

$$= \sqrt[4]{2} \angle \frac{\pi}{8}$$

$$= \frac{\sqrt[4]{2} \sqrt{2 + \sqrt{2}}}{2} + \frac{\sqrt[4]{2} \sqrt{2 - \sqrt{2}}}{2} j$$

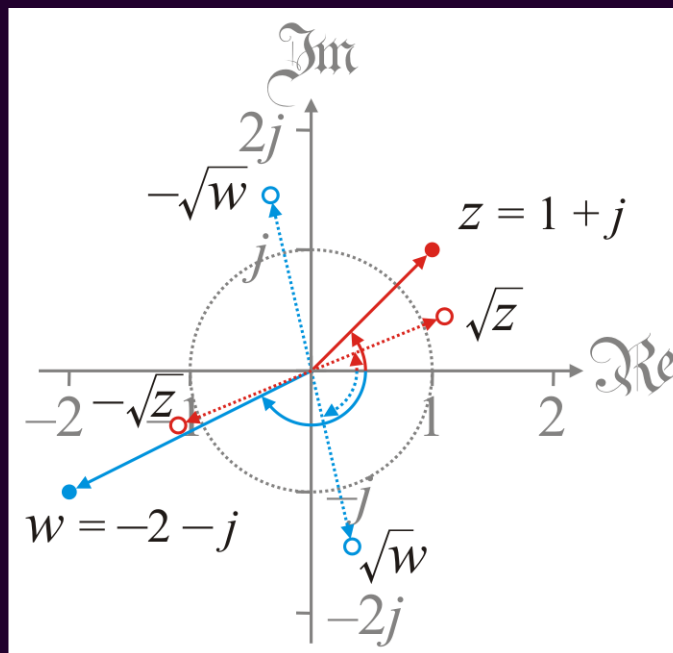
$$\approx 1.0987 + 0.4551j$$





Geometric interpretation

- The geometric interpretation of the square root is
 - The square root of the magnitude
 - Half the angle if $-\pi < \theta \leq \pi$





Properties

Theorem: $|\sqrt{z}| = 1$ if and only if $|z| = 1$

- The square root of a complex number on the unit circle remains on the unit circle

Proof: $\sqrt{|z|} = |z|$ if and only if $|z| = |z|^2$,
which is true if and only if $|z| = 0$ or $|z| = 1$,
but $|z| = 0$ if and only if $z = 0$, so it must be that $|z| = 1$. $\mathcal{QED}$





Summary

- In this topic, we've introduced the square root of a complex number
 - Given any non-zero complex number z ,
 - There are two numbers that when squared equal z
 - We defined one to be principal square root
 - The magnitude is the square root of the magnitude
 - The angle is half the angle if $-\pi < \theta \leq \pi$
 - The square roots of complex numbers on the unit circle remain on that unit circle





References

- [1] https://en.wikipedia.org/wiki/Complex_number#Square_root
- [2] [https://en.wikipedia.org/wiki/Square_root#
Square_roots_of_negative_and_complex_numbers](https://en.wikipedia.org/wiki/Square_root#Square_roots_of_negative_and_complex_numbers)





Acknowledgments

None so far.





Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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for more information.





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